Math 255A Lecture 16 Notes

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1 Compact Operators and Riesz's Theorem

1.1 Compact operators

Last time, we said that a map $T : B_1 \to B_2$ is compact if given $\|x_n\| \leq 1$, then $(Tx_n)_{n \in \mathbb{N}}$ has a convergent subsequence.

Example 1.1. Let $B_1 = C^1([0,1])$ with $\|f\|_{B_1} = \|f\|_{L^\infty} + \|f'\|_{L^\infty}$ and $B_2 = C([0,1])$ eith $\|f\|_{B_2} = \|f\|_{L^\infty}$. Then the inclusion map $B_1 \to B_2$ is compact by Ascoli's theorem.

Example 1.2. Let $k \in C([0,1] \times [0,1])$, and consider $Kf(x) = \int_0^1 k(x,y) f(y)\, dy$. Then $K : L^2((0,1)) \to L^2((0,1))$ is compact by Ascoli's theorem.

Proposition 1.1. *Compact operators have the following properties:*

1. *The space $\mathcal{L}_C(B_1, B_2)$ of compact linear maps $B_1 \to B_2$ is a closed subspace of $\mathcal{L}(B_1, B_2)$.*
2. *Compact operators form an ideal: if $T_1 \in \mathcal{L}(B_1, B_2)$, $T_2 \in \mathcal{L}(B_2, B_3)$, and either T_1 or T_2 is compact, then $T_2 T_2 \in \mathcal{L}_C(B_1, B_3)$.*
3. *If $T \in \mathcal{L}(B_1, B_2)$ has finite rank ($\dim(\operatorname{im}(T)) < \infty$), then T is compact.*

Proof. We prove the properties one at a time:

1. T is compact $\iff$ $T(\{\|x\| \leq 1\})$ compact $\iff$ $\overline{T(\{\|x\| \leq 1\})}$ is complete and totally bounded $\iff$ $T(\{\|x\| \leq 1\})$ is totally bounded. Let $T_n \in \mathcal{L}_C(B_1, B_2)$ be such that $T_n \to T$ in $\mathcal{L}(B_1, B_2)$. Let $\varepsilon > 0$ be given, and let N be such that $\|T - T_N\| < \varepsilon/2$. Then, since $T_N(\{\|x\| \leq 1\})$ is totally bounded, $T_N(\{\|x\| \leq 1\}) \subseteq \bigcup_{j \in I \text{ finite}} B(x_j, \varepsilon/2)$. We get that $T(\{\|x\| \leq 1\}) \subseteq \bigcup_{j \in I} B(x_j, \varepsilon/2)$, so T is compact.
2. This property is clear.
3. We have a factorization $T : B_1 \to B_1/\ker(T) \to B_2$ given by $x \mapsto x + \ker(T) \mapsto Tx$. The space $B_1/\ker(T) \cong \operatorname{im}(T)$ is finite dimensional, and since the identity operator ina finite dimensional space is compact, we get that T is compact. $\square$

1.2 Riesz's Theorem

Theorem 1.1 (F. Riesz)**.** *If the identity map on the Banach space B is compact, then B is finite dimensional.*

Remark 1.1. This is clear if B is a Hilbert space; consider an orthonormal basis.

Lemma 1.1. *Let $B_1 \subsetneq B$ be a proper, closed subspace. Then for every $\varepsilon > 0$, there exists $x \in B$ such that $\|x\| = 1$, and $\operatorname{dist}(x, B_1) = \inf_{y \in B_1} \|x - y\| \geq 1 - \varepsilon$.*

Proof. Let $x \in B \backslash B_1$, and let $d = \operatorname{dist}(z, B_1) > 0$. Let $x_1 \in B_1$ be such that $d \leq \|z - x_1\| < d/(1-\varepsilon)$. We can take $x = (z -_1)/\|z - x_1\|$. For any $y \in B_1$, we have

$$\frac{\|x - y\| = z - x_1 - y\|z - x_1\|\|}{\|z - x_1\|} \geq \frac{d}{\|z - x_1\|} > 1 - \varepsilon. \qquad \square$$

Now we can prove Riesz's theorem.

Proof. If B is infinite-dimensional, there exists a strictly increasing sequence $B_1 \subsetneq B_2 \subsetneq \cdots$ of finite dimensional subspaces of B. Using the lemma, we find $x_j \in B_j$ such that $\operatorname{dist}(x_j, B_{j-1}) \geq 1/2$. In particular, $\|x_j - x_k\| \geq 1/2$ for $k < j$, so (x_j) has no convergent subsequence. $\square$

Theorem 1.2 (Fredholm-Riesz)**.** *Let B be a Banach space, and let $T \in \mathcal{L}_C(B, B)$. Then $I - T$ is Fredholm, and $\operatorname{ind}(I - T) = 0$.*

Before we prove this, let's prove a proposition.

Proposition 1.2. *Let $T \in \mathcal{L}_C(B, B)$. Then*

1. $\dim(\ker(I - T)) < \infty$.
2. $\operatorname{im}(T - T)$ *is closed.*

Proof. This is a crucial observation to any proof of the Fredholm-Riesz theorem.

1. Let $x_n \in \ker(I - T)$ with $\|x_n\| \leq 1$. Then $x_n = Tx_n$ has a convergent subsequence. By Riesz's theorem, $\dim(\ker(I - T)) < \infty$.
2. Let $y \in \overline{\operatorname{im}(I - T)}$ and let $x_n \in B$ be such that $y_n = (1 - T)x_n \to y$. Consider $\operatorname{dist}(x_n, \ker(I - T))$. This equals $\|x_n - z_n\|$ for some $z_n \in \ker(I - T)$ because $y \mapsto \|x_n - y\|$ is continuous and goes to ∞ as $y \to \infty$. We have that $y_n = (I - T)(x_n - z_n) = x_n - z_n - T(x_n - z_n)$.

   We claim that $(x_n - z_n)$ is a bounded sequence. Otherwise, we can assume that $\|x_n \to z_n\| \to \infty$. Let $w_n = (x_n - z_n)/\|x_n - z_n\|$. Then $\|w_n\| = 1$, and $(I - T)w_n =$

$y_n/\|x_n - z_n\| \to 0$ as (y_n) converges. Passing to a subsequence, we may assume that $Tw_n \to v \in B$, so $w_n \to V$. So $(I-T)v = 0$. On the other hand,

$$\mathrm{dist}(w_n, \ker(I-T)) = \frac{\mathrm{dist}(x_n, \ker(I-T))}{\|x_n - z_n\|},$$

so $\mathrm{dist}(v, \ker(I-T)) \leq 1$, and we get the claim. We will finish the proof next time. $\square$